Hamiltonian averaging in soliton-bearing systems with a periodically varying dispersion

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Optical pulse dynamics in dispersion-managed fiber lines is studied using a combination of a Lagrangian approach and Hamiltonian averaging. By making self-similar transform in the Lagrangian and assuming in the leading order a bell-shaped pulse dynamics, we reduce the original system to a nonautonomous Hamiltonian system with two variables. Subsequent Hamiltonian averaging gives a function of two variables whose extrema correspond to periodic pulses. To describe a fine structure of the pulse tails, we further develop Hamiltonian averaging using the complete set of Gauss-Hermite functions and also applying averaging in the spectral domain. [S1063-651X(99)50504-7]

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I. INTRODUCTION

Impressive progress in soliton-based optical data transmission has clearly shown how the fundamental soliton theory can be successfully exploited in important practical applications such as high-bit-rate optical communications. Practical achievements have stimulated further studies of soliton dynamics in media with varying coefficients. In this Rapid Communication, we present general Hamiltonian approaches to describe average envelope soliton propagation in a medium with large periodic variations of dispersion. As a specific practical application, we focus here on dispersion-managed (DM) soliton transmission. The traditional path-averaged optical soliton preserves its shape during propagation by compensating the fiber dispersion through nonlinearity; only the pulse power oscillates due to periodic amplification of the pulse to compensate for the fiber loss. Rapid oscillations of the power can be averaged out and, as a result, the slow pulse dynamics in the traditional transmission lines with constant dispersion is governed by the nonlinear Schrödinger equation (NLSE). The DM soliton that occurs in the system with large variations of the dispersion differs substantially from the fundamental soliton [1–18]. There are two scales in the DM systems: the first (fast dynamics) corresponds to rapid oscillations of the pulse width and power due to periodic variations in the dispersion and periodic amplification; the second (slow dynamics) occurs due to the combined effects of nonlinearity, residual dispersion and pulse chirping. To describe slow dynamics of the DM soliton one should average the propagation equation over fast oscillations. The small parameter in the problem (and throughout the paper) is $\epsilon = L/Z_{NL}$, with L as a compensation period and Z_{NL} as a characteristic nonlinear scale. Overall, the pulse dynamics in these systems is rather complicated and typically depends on many system parameters. Different theoretical approaches have been developed to describe properties of DM soliton, which include interesting numerical methods [1–4], a variational approach [5–11], a root-mean-square momentum method [11,18], multiscale analysis [12–14], different averaging methods [15,16], including averaging in the spectral domain [5,6], and an expansion of the DM soliton on the basis of the chirped Gauss-Hermite functions [16,17]. Because of the practical importance of the problem, it is of evident interest to develop different analytical methods to describe the properties of the DM soliton. A variety of complimentary mathematical methods can be advantageously used to find an optimal and economical description of any specific practical application. In this paper we present an averaging approach based on a combination of the Lagrangian approach and Hamiltonian averaging.

II. BASIC EQUATIONS IN THE LAGRANGIAN FORM AND THE TWO-PDE APPROXIMATION

The optical pulse dynamics is governed by the following partial differential equation (PDE) with the periodic coefficients $d(z)$ and $c(z)$ (we assume here that both have the same period).

$$iA_z + d(z)A_{tt} + \epsilon c(z)|A|^2 A = 0, \tag{1}$$

where we are using notations of [11], but setting aside a small parameter ϵ in $c(z)$ with new $c(z)$ to be of the order of one, and arbitrary $d(z) = \tilde{d} + \langle d \rangle$ ($\langle \tilde{d} \rangle = 0$). The distance is normalized by the compensation period L. The action integral of this equation is given by

$$\Xi=\int dt\,dz\left\{\frac{i}{2}(A\bar{A}_z-\bar{A}A_z)+d(z)|A_t|^2-\epsilon\frac{c(z)}{2}|A|^4\right\}.$$

By introducing the transformation from A to Q,

$$A=\frac{N}{\sqrt{T(z)}}Q(\xi,z)\exp\left[i\frac{M(z)}{T(z)}t^2\right],\quad \xi=\frac{t}{T(z)},$$

which accounts for the fast (dominant) self-similar dynamics, we obtain a new action integral:

$$\Xi=\int dt\,dz\left\{\frac{i}{2}\left(-\frac{tT_z}{T^3}[QQ^*_\xi-Q^*Q_\xi]+\frac{1}{T}[QQ^*_z-Q^*Q_z]\right)+\frac{t^2}{T}|Q|^2\left(\frac{M}{T}\right)_z-\epsilon\frac{N^2c(z)}{2T^2}|Q|^4+\frac{d(z)}{T^3}(|Q_\xi|^2+4M^2t^2|Q|^2+i2Mt[QQ^*_\xi-Q^*Q_\xi])\right\}. \tag{2}$$

Assuming that $\partial_\xi Arg(Q)=0$ and introducing new integration variables, we obtain an averaged action (here $C_1=\int|Q_\xi|^2d\xi/\int|Q|^2d\xi$ and $C_2=\int|Q|^4d\xi/\int|Q|^2d\xi$)

$$\bar{\bar{\Xi}}=\int dz\left\{d(z)\left(\frac{C_1}{T^2}+4M^2\right)-\epsilon\frac{N^2c(z)C_2}{2T}-2MT_z\right\}.$$

Now, observing that the Lagrangian has the form $L(M,T)=H(M,T)-2MT_z$ which can be easily rescaled to $L(p,q)=H(p,q)-p\dot{q}$ we get for p and q the Hamiltonian system [7] with the Hamiltonian

$$H=d(z)\left[\frac{p^2}{2}+\frac{1}{2q^2}\right]-\epsilon\frac{N^2c(z)}{q}.$$

III. HAMILTONIAN AVERAGING

We will carry out the averaging procedure while preserving the Hamiltonian structure. The minima and maxima of the averaged Hamiltonian function will correspond to the fixed points. Consider the Hamiltonian $H=d(z)H_0(p,q)+\epsilon H_1(p,q,z)$, which is integrable if $\epsilon=0$. Introducing new variables (I,ξ), where $I=H_0(p,q)$ and ξ is chosen so as to keep the Hamiltonian structure, we obtain the new Hamiltonian $H=d(z)I+\epsilon\tilde{H}_1(I,\xi,z)$, which takes the form $H=\epsilon\tilde{H}_1(I,\eta+R_0(z),z)$ after the transformation $\xi=R_0(z)+\eta$ $(dR_0/dz=\tilde{d}(z))$. The critical points of the averaged Hamiltonian $\bar{H}(I,\eta)$ give the first approximation to the fixed points. We now compute the above quantities for the considered problem. We look for the transformation given implicitly by

$$q=\partial_pS(I,p),\quad \xi=-\partial_IS(I,p), \tag{3}$$

where $S(I,p)$ is a generating function, so that $2I=p^2+q^{-2}$. Solving the above equality for q, $q=1/\sqrt{2I-p^2}$, and using the first equation in Eqs. (3), we obtain a generating function $S(I,p)=\arcsin[p/\sqrt{2I}]$. Then the second equation in Eqs. (3) gives the relation $\xi=p/(2I\sqrt{2I-p^2})$. Now, solving the equations for p and q, we obtain

$$p^2=2I\frac{(2I\xi)^2}{1+(2I\xi^2)},\quad q^2=\frac{1+(2I\xi)^2}{2I}. \tag{4}$$

The transformed Hamiltonian takes the form

$$H(I,\xi,z)=d(z)I-\epsilon N^2c(z)\frac{\sqrt{2I}}{\sqrt{1+(2I\xi)^2}}.$$

Carrying out the transform $\xi=\eta+R_0(z)$ we finally get

$$H(I,\eta,z)=\langle d\rangle I-\epsilon N^2c(z)\frac{\sqrt{2I}}{\sqrt{1+[2I(\eta+R_0(z))]^2}}.$$

The problem is now reduced to the search for minima and maxima of the averaged Hamiltonian function

$$\bar{H}(I,\eta)=\langle d\rangle I-\epsilon N^2\int_0^1\frac{\sqrt{2I}c(z)dz}{\sqrt{1+[2I(\eta+R_0(z))]^2}}. \tag{5}$$

Note that the last integral in Eqs. (4) can be evaluated explicitly [18] if $c(z)=$const and d is a piecewise constant.

IV. HAMILTONIAN AVERAGING OF THE PDE

The previous analysis proves to be useful in determining the averaged quantities that relate to the width and pulse chirp, without determining the pulse profile. This must be assumed to compute the constants C_1,C_2, which arise in the formula for the averaged action $\bar{\bar{\Xi}}$. In this section we take the next step in our analysis by averaging the PDE in the general case, while keeping in the leading order the self-similar structure obtained in the preceding section. To do this, we expand (see [16]) a function $Q(\xi,z)$ using a complete set of orthogonal chirped Gauss-Hermite functions $Q(\xi,z)=\Sigma_nb_n(z)f_n(\xi)$ with $(f_n)_{\xi\xi}-\xi^2f_n=\lambda_nf_n$, $\lambda_n=-1-2n$, where normalized $f_n(x)=\exp(-x^2/2)H_n(x)/(\sqrt{2^nn!\sqrt{\pi}})$. Here $H_n(x)$ is the nth-order Hermite polynomial and the coefficients $b_n(z)$ in the expansion can be found by scalar multiplication with f_n. Because the dependence of $Q(\xi,z)$ on ξ is now determined by known functions $f_n(\xi)$, after integrating over ξ the Lagrangian L in $\Xi=\int L\,dz$ is

$$L=\sum_{n=0}^{\infty}\left(\frac{i}{2}\left[b_n\frac{db_n^*}{dz}-b^*\frac{db_n}{dz}\right]-\frac{d(z)}{T^2}\lambda_n|b_n|^2\right)+\sum_{n,m}^{\infty}\left[T^2\frac{d}{dz}\left(\frac{M}{T}\right)+4M^2d(z)-\frac{d(z)}{T^2}\right]b_nb_m^*S_{n,m}-\epsilon\frac{N^2c(z)}{2T}\sum_{n,m,l,k}b_nb_mb_l^*b_k^*V_{n,m,l,k}+\frac{i}{2}[T_z-4Md(z)]\sum_{n=0}^{\infty}(b_nb_{n+2}^*-b_{n+2}b_n^*)(n+2). \tag{6}$$

$S_{n,m}=\int_{-\infty}^{+\infty}f_nx^2f_mdx$, $V_{m,l,k,m}=\int_{-\infty}^{+\infty}f_nf_mf_lf_kdx$. Any $S_{n,m}$ and $V_{m,l,k,n}$ can be found in explicit form [16]. Note that the derived exact expression is reduced to the Lagrangian from the previous section under the assumption $b_n(z)=\text{const}$ and $\Sigma_{n=0}^{\infty}(b_n b_{n+2}^* - b_{n+2}^* b_n)\ (n+2)=0$ [which corresponds to the condition $\partial_\xi Arg(Q)=0$]. Higher order non-self-similar corrections to the DM pulse dynamics are accounted through the functions b_n. To preserve in the leading order the self-similar structure of the pulse core, we now choose the functions $T(z)$ and $M(z)$ to be periodic solutions of

$$\frac{dT}{dz}=4d(z)M;\quad \frac{dM}{dz}=\frac{d(z)}{T^3}-\epsilon\frac{c(z)N^2}{T^2}. \tag{7}$$

In this case, the Lagrangian L takes the form

$$L=\sum_{n=0}^{\infty}\left(\frac{i}{2}\left[b_n\frac{db_n^*}{dz}-b_n^*\frac{db_n}{dz}\right]-\frac{d(z)}{T^2}\lambda_n|b_n|^2\right)$$
$$-\frac{\epsilon N^2 c}{T}\sum_{n,m}^{\infty} b_n b_m^* S_{n,m}$$
$$-\frac{\epsilon N^2 c}{2T}\sum_{n,m,l,k} b_n b_m b_l^* b_k^* V_{n,m,l,k}.$$

This means that we have reduced the periodic problem for the original PDE (1) to an infinite set of ordinary differential equations with periodic boundary conditions, determining the dynamics of b_n. The fast decay of b_n in n makes this basis very useful in practical applications. By introducing the notation $b_n=F_n\exp[-i\Theta_n]=\sqrt{p_n}\exp[-iq_n]$, the Lagrangian can be written in a usual form $L(q_1,p_1,q_2,p_2,\ldots,z)=H(q_1,p_1,q_2,p_2,\ldots,z)-\Sigma_n\dot{q}_n p_n$, where

$$-H=\sum_{n=0}\frac{d(z)}{T^2}\lambda_n p_n+\frac{\epsilon N^2 c}{T}\sum_{n,m}^{\infty}\sqrt{p_n p_m}S_{n,m}\cos(q_m-q_n)$$
$$+\frac{\epsilon N^2 c}{2T}\sum_{n,m,l,k}\sqrt{p_n p_m p_l p_k}V_{n,m,l,k}$$
$$\times\cos(q_l+q_k-q_m-q_n).$$

Applying the procedure developed in the previous section we present the Hamiltonian as

$$H(q_1,p_1,q_2,p_2,\ldots,z)$$
$$=H(q,p)=\frac{d(z)}{T^2}H_0(q,p)+\epsilon H_1(q,p,z)$$
$$=-\frac{d(z)}{T^2}\sum_{n=0}\lambda_n p_n+\epsilon H_1(q,p,z),$$

which is integrable if $\epsilon=0$. Namely, if $\epsilon=0$, $p_n=\text{const}$ and $q_n=-\lambda_n\int^z d(s)ds/T^2(s)$. After the transformation $q_n=-\lambda_n R(z)+\eta_n$ with R defined from $dR/dz=d(z)/T^2-\langle d/T^2\rangle$, the Hamiltonian takes the form

$$H=-\left\langle\frac{d(z)}{T^2}\right\rangle\sum_{n=0}\lambda_n p_n+\epsilon\tilde{H}_1(p_n,\eta_n-\lambda_n R(z),z).$$

Extrema of this function give a profile of the DM soliton. In many practical problems it is enough to take into account only the few first terms in the above expansion of the DM pulse in the basis of the Gauss-Hermite functions.

V. AVERAGING OF THE PDE IN THE FREQUENCY DOMAIN

In this section we present an alternative way to average Eq. (1) by going into the frequency domain. The averaged equation in the frequency domain has been derived in [5,6]. This approach allows one to decompose DM pulse dynamics in the fast evolution of the phase and a slow evolution of the amplitude. The shape of the DM soliton is then given by a nonlocal integral equation. We present here a compact form of the averaged equation that we hope could be useful for practical numerical simulations.

Following [5,6] we take the Fourier transform in Eq. (1) and account for the fast oscillations of the phase as

$$A(t,z)=\int_{-\infty}^{+\infty}d\omega\, q(\omega,z)\exp[-i\omega t-i\omega^2 R_0(z)]; \tag{8}$$

here $dR_0(z)/dz=\tilde{d}(z)$ and $\langle R_0\rangle=0$. The aim of this transformation is to eliminate the large coefficient $\tilde{d}$ from Eq. (1). In the new variables the propagation equation takes the form $iq_z=\delta H/\delta q^*$ with the Hamiltonian H given by

$$H=\langle d\rangle\int_{-\infty}^{\infty}\omega^2|q(\omega,z)|^2 d\omega-\epsilon\frac{c(z)}{2}\int_{-\infty}^{\infty}d\omega_1 d\omega_2 d\omega_3 d\omega_4$$
$$\times\delta(\omega_1+\omega_2-\omega_3-\omega_4)e^{iR_0\Delta\Omega}q_{\omega_1}q_{\omega_2}q_{\omega_3}^* q_{\omega_4}^*;$$

here $\Delta\Omega=\omega_1^2+\omega_2^2-\omega_3^2-\omega_4^2$. To get rid of the δ function we introduce the linear change of variables $2l_1=\omega_2+\omega_1$, $2l_2=\omega_4+\omega_3$, $2m_1=\omega_2-\omega_1$, $2m_2=\omega_4-\omega_3$; then the Hamiltonian takes the form

$$H=\langle d\rangle\int_{-\infty}^{\infty}m^2|q(m,z)|^2 dm-\epsilon c(z)\int_{-\infty}^{\infty}dl\, dm_1 dm_2$$
$$\times e^{i2R_0(m_1^2-m_2^2)}q(l-m_1)q(l+m_1)q^*(l-m_2)q^*$$
$$\times(l+m_2).$$

Though the main results here are formulated in a general form and can be used for arbitrary dispersion maps, we illustrate the procedure by considering a two-step lossless ($c=1$) map: $d(z)=d_1+\langle d\rangle$ if $0<z<L_1$, and $d(z)=d_2+\langle d\rangle$ for $L_1<z<L$. Here $d_1L_1+d_2(L-L_1)=0$ and parameter $\mu=d_1L_1$ is a characteristic of the strength of the map. Since $q(\omega,z)$ varies slowly, in the first approximation we can integrate over the period the above equation placing q outside of the integrals in z [5,6]. This makes it possible [5,6] to explicitly express a field at the end of the section as a function of the input field (i.e., to solve the mapping problem) and to describe an average dynamics. Carrying out this procedure we get a Hamiltonian integrodifferential equation in the spectral domain describing the slow evolution of $q(\omega,z)$ [5,6],

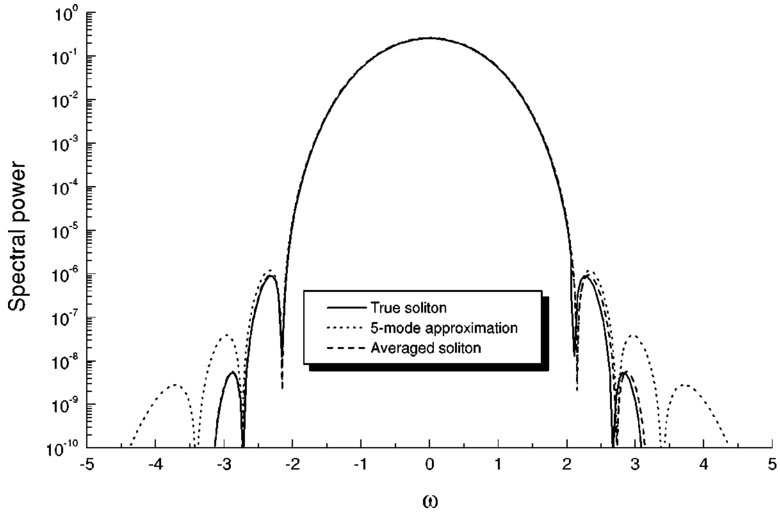

FIG. 1. Shape of the DM soliton is shown in the logarithmic scale: true periodic solution of Eq. (1) (solid line), solution of the averaged Eq. (9) (dashed line), and five-mode approximation based on the Gauss-Hermite expansion (dotted line). Here $d(z)=\pm 5+0.15$.

$$i\frac{\partial q(z,\omega)}{\partial z}=\frac{\delta H}{\delta q^*}=\omega^2\langle d\rangle q(z,\omega)-\epsilon\int d\omega_1 d\omega_2\frac{\sin[\mu(\omega-\omega_1)(\omega-\omega_2)]}{\mu(\omega-\omega_1)(\omega-\omega_2)}\times q_{\omega_1}q_{\omega_2}+q^*_{\omega_1+\omega_2-\omega},$$

with the Hamiltonian

$$H=\langle d\rangle\int\omega^2|q|^2d\omega-\epsilon\int dl\,dm_1dm_2q(l+m_1)\times q(l-m_1)q^*(l+m_2)q^*(l-m_2)\frac{\sin[\mu(m_1^2-m_2^2)]}{\mu(m_1^2-m_2^2)}.$$

In the limit $\mu=0$ we just get the Fourier transform of the NLSE. The soliton in this case has the well-known sech shape. The general DM soliton solution has the form $q(\omega,z)=\exp(ikz)F(\omega)$ with the shape $F(\omega)$ given by the equation

$$(k+\omega^2\langle d\rangle)F(\omega)=\epsilon\int d\omega_1 d\omega_2\frac{\sin[\mu(\omega-\omega_1)(\omega-\omega_2)]}{\mu(\omega-\omega_1)(\omega-\omega_2)}\times F_{\omega_1}F_{\omega_2}F^*_{\omega_1+\omega_2-\omega}. \quad (9)$$

A typical solution of this equation (and comparisons with numerics and with the expansion using Gauss-Hermite functions) is presented in Fig. 1. Note that the solution to this equation does exist in the region of zero and normal ($\langle d\rangle<0$), in agreement with the observations in [2].

In conclusion, we have developed Hamiltonian averaging to describe the slow (average) dynamics of the dispersion-managed optical pulse in fiber transmission lines. Derived equations described both the Gaussian core and the exponentially decaying oscillating tails of DM soliton.

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